TWO POINT PROBLEMS AND ANALYTICITY OF SOLUTIONS OF ABSTRACT PARABOLIC EQUATIONS[†]

BY

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ABSTRACT

For $t \in [a, b]$, let $A(t)$ be the unbounded operator in $H^{0,p}(G)$ associated with an elliptic-boundary value problem that satisfies Agmon's conditions on the rays $\lambda = \pm i\tau, \tau \geqq 0$. Existence and uniqueness results are obtained for weak and strict solutions of two-point problems of the type $(du/dt) - A(t)u(t) = f(t)$, $E_1(\alpha)u(\alpha) = u_\alpha$, $E_2(\beta)u(\beta) = u_\beta$. Here $[\alpha, \beta) \subseteq [a, b]$, $E_1(\alpha)$ and $E_2(\beta)$ are spectral projections associated with $A(\alpha)$ and $A(\beta)$ respectively, and $A(\alpha)E_1(\alpha)$ and $= A(\beta)E_2(\beta)$ are infinitesimal generators of analytic semigroups. When $A(t)$ and $f(t)$ are analytic in a convex, complex neighborhood O of $[a, b]$ we show that for some θ_i, $i = 1,2$, any solution of $du/dt = A(t)u(t) = f(t)$ in $[a, b]$ is analytic and satisfies the above equation in the set $O \cap \{t; t \neq a, t \neq b, |\arg(t-a)| < \theta_1, |\arg(b-t)| < \theta_2\}$.

1. Introduction

Consider the abstract parabolic equation

$$\frac{du}{dt} - A(t)u(t) = f(t) \tag{1.1}$$

where for $t \in [a, b]$, $A(t)$ is the unbounded operator in $H^{0,p}(G)$ associated with an elliptic boundary value problem that satisfies Agmon's conditions on the rays $\lambda = i\tau$ and $\lambda = -i\tau$, $\tau \geqq 0$ (see [1] and [9]). We introduce in this work a class of two-point problems for (1.1) that are well posed for sufficiently small subintervals $[\alpha, \beta]$ of $[a, b]$. Existence and uniqueness results for weak and strict solutions of such problems are obtained.

It is shown that any solution $u(t)$ of (1.1) in $[\alpha, \beta]$ satisfies an inequality of the type

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$$\| u(t) \| \leqq C\left(\| E_1(\alpha)u(\alpha) \| + \| E_2(\beta)u(\beta) \| + \max_{t \in [\alpha,\beta]} \| f(t) \| \right)$$

provided that $\beta-\alpha$ is small enough. $E_1(\alpha)$ and $E_2(\beta)$ are certain spectral projection associated with the operators $A(\alpha)$ and $A(\beta)$ respectively.

When $A(t)$ and $f(t)$ are analytic in a convex, complex neighborhood O of $[a,b]$ we show that for some θ_i, $i = 1,2$, any solution of (1.1) in $[a,b]$ is analytic and satisfies the equation (1.1) in the set $O \cap \{t; t \neq a, t \neq b, |\arg(t-a)| < \theta_1, |\arg(b-t)| < \theta_2\}$. For $A(t)$ independent of t, this result follows from the results of [2]. For $p = 2$ the analyticity of $u(t)$ in (a,b) follows from [6] and [10].

2. Notation

Given two Banach spaces X and Y, we denote by $B(X,Y)$ the space of bounded linear operators from X to Y. X^* is the adjoint space of X. The domain of a closed and densely defined linear operator A in X is denoted by $D(A)$. $\rho(A)$ is the resolvent set of A and $\sigma(A)$ is the spectrum of A. The norm of an element $u \in X$ is denoted by $\| u \|_X$, and when X is fixed, by $\| u \|$. For $k = 0,1,\cdots$, $C^k([a,b],X)$ is the space of k times continuously differentiable functions from the interval $[a,b]$ to X. For $u(t) \in C^k([a,b],X)$ and $j = 0,\cdots,k$, $|u(t)|_j = \max_{t \in [a,b]} \| (d^j/dt^j)u(t) \|_X$. $C([a,b],X) = C^0([a,b],X)$ and $C^\infty([a,b],X) = \bigcap_{k=0}^{\infty} C^k([a,b]X)$. When X is fixed we set $C^k[a,b] = C^k([a,b],X)$ and $C^\infty[a,b] = C^\infty([a,b],X)$. For $k = 0,1,\cdots$ and $\alpha > 0$ $C^k_\alpha([a,b],X)$ is the subset of elements $u(t)$ of $C^k([a,b],X)$ for which there exists a constant C such that for t_1, $t_2 \in [a,b]$ we have $\| (d^k/dt^k)u(t_1) - (d^k/dt^k)u(t_2) \| \leqq C|t_1 - t_2|^\alpha$.

Let G be a bounded domain in R^ν with boundary ∂G of class C^∞. Denote by $C^\infty(\bar{G})$ the set of l tuples of infinitely differentiable complex-valued functions in $\bar{G}$, the closure of G. For $1 < p < \infty$ and $\omega = 0,1,\cdots$, $H^{\omega,p}(G)$ is the completion of $C^\infty(\bar{G})$ under the norm $\sum_{|\alpha| \leqq \omega} (\int |D^\alpha f(x)|^p dx)^{1/p}$. We use the standard notation $x = (x_1,\cdots,x_\nu)$, $D_j = i(\partial/\partial x_j)$, $D = (D_1,\cdots,D_\nu)$ and $D^\alpha = D_1^{\alpha_1}\cdots D_\nu^{\alpha_\nu}$. $\alpha = (\alpha_1,\cdots,\alpha_\nu)$ is a multi-index of non-negative integers and $|\alpha| = \sum_{i=1}^{\nu} \alpha_i$.

3. Theorems and proofs

For $t \in [a,b]$, let $A(t)$ be a closed and densely defined linear operator in a Banach space X. We assume, in the rest of the paper that the following conditions are satisfied.

Condition I. For $i = 1,2$ and $t \in [a,b]$, $E_i(t)$ is a bounded projection in X

and $A(t)$ is completely reduced by the direct sum decomposition $X = \sum_{i=1}^{2} \oplus E_i(t)X$.

Condition II. $E_i(t) \in C^1([a,b], B(X,X))$ for $i = 1,2$.

Condition III. There exist complex numbers μ_i, $i=1,2$, such that for $i = 1,2$, the operator $L_i(t) = (-1)^{i+1}(A(t)E_i(t) + \mu_i I)$ satisfies the three conditions:

(i) For $t \in [a, b]$, the resolvent set of $L_i(t)$ contains the closed sector $\Gamma(-\frac{1}{2}\pi - \theta, \frac{1}{2}\pi + \theta)$ with $0 < \theta < \frac{1}{2}\pi$.

(ii) $L_i(t)^{-1} \in C^1([a,b], B(X,X))$.

(iii) There exist constants C_j such that for $j = 0,1, \lambda \in \Gamma(-\frac{1}{2}\pi - \theta, \frac{1}{2}\pi + \theta)$ and $t \in [a,b]$ we have

$$\left\| \frac{\partial^j}{\partial t^j}(\lambda - L_i(t))^{-1} \right\| \leq \frac{C_j}{|\lambda|}. \tag{3.1}$$

DEFINITION 3.1. Let $f(t) \in C[a,b]$. Suppose that $u_a \in E_1(a)X$ and that $u_b \in E_2(b)X$. We say that $\phi(t)$ is a test function for the two-point problem

$$\frac{du}{dt} - A(t)u(t) = f(t), \tag{3.2}$$

$$E_1(a)u(a) = u_a, \tag{3.3}$$

$$E_2(b)u(b) = u_b \tag{3.4}$$

if the following conditions are satisfied:

(i) $\phi(t) \in D(A(t)^*)$ for $t \in [a,b]$,

(ii) $\phi(t) \in C^1([a,b], X^*)$ and $A(t)^*\phi(t) \in C([a,b], X^*)$,

(iii) $E_2(a)^*\phi(a) = 0$ and $E_1(b)^*\phi(b) = 0$.

We say that $u(t)$ is a weak solution of (3.2), (3.3), and (3.4) if $u(t) \in C[a,b]$ and

$$\int_a^b (u(t), \phi'(t) + A(t)^*\phi(t))\, dt + \int_a^b (f(t), \phi(t))dt + (u_a, \phi(a)) - (u_b, \phi(b)) = 0 \tag{3.5}$$

for every test function $\phi(t)$ of (3.2), (3.3), and (3.4).

We say that $u(t)$ is a strict solution of (3.2), (3.3), and (3.4) if

$$u(t) \in C[a,b] \cap C^1(a,b)$$

for $t \in (a,b)$, $u(t) \in D(A(t))$, and equation (3.2) holds; and the relations (3.3) and (3.4) are satisfied.

We shall use in the rest of the paper the following results of Kato and Tanabe

[7]. Suppose that $L(t)$ satisfies (i), (ii), and (iii) of Condition III. Assume that $B(t) \in C([a,b], B(X,X))$. Then the initial value problem

$$\frac{du}{dt} = (L(t) + B(t))u(t) + f(t), \tag{3.6}$$

$$u(a) = u_a \tag{3.7}$$

has a unique weak solution for every $u_a \in X$. This solution is given by $u(t) = U(t,a)u_a + \int_a^t U(t,\tau)f(\tau)d\tau$ where $U(t,\tau)$ is the evolution operator associated with $L(t) + B(t)$. This result is proved in [7] when $B(t) \equiv 0$ and can be verified in the general case using the methods of [7]. It is shown in [6] that the above-mentioned solution $u(t)$ is a strict solution provided that

$L(t)^{-1} \in C_\alpha^1([a,b], B(X,X))$, $B(t) \in C_\beta([a,b], B(X,X))$,

and $f(t) \in C_\gamma[a,b]$ for some positive constants α, β, and γ.

DEFINITION 3.2. Assume that $A(t)$ satisfies Conditions I, II, and III. For $i = 1, 2$, let $B_i(t) = \sum_{j=1}^2 E_j'(t)E_j(t) - \mu_i I$. Let $K_1(t,\tau)$, $a \leqq \tau \leqq t \leqq b$, be the evolution operator associated with $L_1(t) + B_1(t)$. Let $H(t,\tau)$, $a \leqq \tau \leqq t \leqq b$, be the evolution operator associated with $L_2(a+b-t) - B_2(a+b-t)$, and for $a \leqq t \leqq \tau \leqq b$ set $K_2(t,\tau) = H(a+b-t,\, a+b-\tau)$. Define $W_1(t,\tau)$ for $a \leqq \tau \leqq t \leqq b$, and $W_2(t,\tau)$ for $a \leqq t \leqq \tau \leqq b$ by

$$W_i(t,\tau) = K_i(t,\tau)E_i(\tau)E_i'(\tau). \tag{3.8}$$

LEMMA 3.3. *Assume that $A(t)$ satisfies Conditions* I, II, *and* III. *Then $u(t)$ is a weak solution of* (3.2), (3.3), *and* (3.4) *and, for $i = 1,2$, $u_i(t) = E_i(t)u(t)$ if and only if $u_i(t) \in C[a,b]$ for $i = 1, 2$ and*

$$u_1(t) = K_1(t,a)u_a + \int_a^t K_1(t,\tau)E_1(\tau)f(\tau)d\tau + \int_a^t W_1(t,\tau)u_2(\tau)d\tau, \tag{3.9}$$

$$u_2(t) = K_2(t,b)u_b + \int_t^b K_2(t,\tau)E_2(\tau)f(\tau)d\tau + \int_t^b W_2(t,\tau)u_1(\tau)d\tau. \tag{3.10}$$

PROOF. It follows, from the remarks preceding this lemma and from the definition of $K_i(t,\tau)$ and $W_i(t,\tau)$ for $i = 1,2$, that $u_i(t) \in C[a,b]$ for $i = 1,2$, and (3.9) and (3.10) hold if and only if $u_1(t)$ is a weak solution of

$$\frac{du_1}{dt} = \left(A(t)E_1(t) + \sum_{i=1}^{2} E_i'(t)E_i(t)\right)u_1(t) + E_1(t)(f(t) + E_1(t)u_2'(t)). \tag{3.11}$$

$$u_1(a) = u_a, \tag{3.12}$$

and $u_2(t)$ is the weak solution of

$$\frac{du_2}{dt} = (A(t)E_2(t) + \sum_{i=1}^{2} E_i(t)'E_i(t))u_2(t) + E_2(t)(f(t) + E_2'(t)u_1(t)), \tag{3.13}$$

$$u_2(b) = u_b. \tag{3.14}$$

Let $u(t)$ be a weak solution of (3.2), (3.3), and (3.4) and let $\psi(t)$ be a test function of (3.11) and (3.12). Since $(A(t)E_1(t))^* = A(t)^*E_1(t)^*$ we have $\psi(t) \in D(A(t)^*E_1(t)^*)$ for $t \in (a,b)$. Also $\psi(t) \in C^1([a,b],X^*)$, $A(t)^*E_1(t)^*\psi(t) \in C([a,b],X^*)$, and $\psi(b) = 0$. Consequently $E_1(t)^*\psi(t)$ is a test function for the two-point problem (3.2), (3.3), and (3.4) and

$$\int_a^b (u(t), (E_1(t)^*\psi(t))' + A(t)^*E_1(t)^*\psi(t))dt + \int_a^b (f(t), E_1(t)^*\psi(t))dt + (u_a, E_1(a)^*\psi(a)) = 0. \tag{3.15}$$

It follows from (3.15) that

$$\int_a^b (E_1(t)\ u(t), \psi(t)' + (A(t)^*E_1(t)^* + \sum_{i=1}^{2} E_i(t)^*E_i(t)^{*\prime})\psi(t))dt + \int_a^b (E_1(t)f(t) + E_1'(t)E_2(t)u(t), \psi(t))\,dt + (u_a, \psi(a)) = 0. \tag{3.16}$$

(3.16) implies that $E_1(t)u(t)$ is a weak solution of (3.11) and (3.12). One verifies similarly that $E_2(t)u(t)$ is a weak solution of (3.13) and (3.14) and this implies that (3.9) and (3.10) hold.

Before proving the second part of this lemma we make the following observation. Let $g(t) \in C[a,b]$. Suppose that $v_a \in E_1(a)X$ and that $v(t)$ is the weak solution of the initial value problem

$$\frac{dv}{dt} = \left(A(t)E_1(t) + \sum_{i=1}^{2} E_i'(t)E_i(t)\right)v(t) + E_1(t)g(t), \tag{3.17}$$

$$v(a) = v_a. \tag{3.18}$$

Then $E_1(t)v(t) = v(t)$ for $t \in [a,b]$. To verify this assertion note that for every test function $\phi(t)$ of (3.17) and (3.18), $E_1(t)^*\phi(t)$ is a test function for the same problem and

$$\int_a^b \left(v(t), (E_1(t)^*\phi(t))' + (A(t)^*E_1(t)^* + \sum_{i=1}^{2} E_i(t)^*E_i(t)^{*\prime})\, E_1(t)^*\phi(t)\right)dt + \int_a^b (E_1(t)g(t), E_1(t)^*\phi(t))dt + (v_a, E_1(a)^*\phi(a)) = 0. \tag{3.19}$$

Since

$$E_1(t)^{*\prime} + \left(\sum_{i=1}^{2} E_i(t)^* E_i(t)^{*\prime}\right) E_1(t)^* = E_1(t)^* \left(\sum_{i=1}^{2} E_i(t)^* E_i(t)^{*\prime}\right)$$

it follows from (3.18) that

$$\int_a^b (E_1(t)v(t), \phi(t)' + \left(A(t)^* E_1(t)^* + \sum_{i=1}^{2} E_i(t)^* E_i(t)^{*\prime}\right)\phi(t))dt$$
(3.20)
$$+ \int_a^b (E_1(t)g(t), \phi(t))\, dt + (v_a, \phi(a)) = 0.$$

Consequently $E_1(t)v(t)$ is a weak solution of (3.17) and (3.18) and the uniqueness of the weak solution of (3.17) and (3.18) implies that $v(t) = E_1(t)v(t)$ for $t \in [a, b]$. One verifies similarly that if $v_b \in E_2(b)X$ and if $v(t)$ is a weak solution of

$$\frac{dv}{dt} = \left(A(t)E_2(t) + \sum_{i=1}^{2} E_i'(t)E_i(t)\right)v(t) + E_2(t)g(t), \tag{3.21}$$

$$v(b) = v_b, \tag{3.22}$$

then $E_2(t)v(t) = v(t)$ for $t \in [a, b]$.

Let $u_i(t) \in C[a, b]$ for $i = 1,2$. Suppose that $u_1(t)$ is a weak solution of (2.11) and (2.12), and that $u_2(t)$ is a weak solution of (3.13) and (3.14). The last observation implies that $E_i(t)u_i(t) = u_i(t)$ for $i = 1,2$ and $t \in [a, b]$. Let $\phi(t)$ be a test function for (3.2), (3.3), and (3.4). Then $E_1(t)^*\phi(t)$ is a test function for (3.11) and (3.12) and

$$\int_a^b \left(u_1(t), (E_1(t)^*\phi(t))' + (A(t)^* E_1(t)^* + \sum_{i=1}^{2} E_i(t)^* E_i(t)^{*\prime})\, E_1(t)^*\phi(t) \right) dt$$
(3.23)
$$+ \int_a^b (E_1(t)E_1(t)' u_2(t), E_1(t)\phi(t))dt + \int_a^b (E_1(r)f(t), E_1(t)^*\phi(t))dt$$
$$+ (u_a, E_1(a)^*\phi(a)) = 0.$$

Since $E_1(t)^* E_1(t)^{*\prime} E_1(t)^* = 0$ and $u_i(t) = E_i(t)u(t)$ for $i = 1,2$, it follows from (3.23) that

$$\int_a^b (u_1(t), \phi(t)' + (A(t)^* + E_1(t)^{*\prime})\phi(t))dt + \int_a^b (E_1(r)' u_2(t), \phi(t))dt$$
(3.24)
$$+ \int_0^b (E_1(t)f(t), \phi(t))\, dt + (u_a, \phi(a)) = 0.$$

One verifies similarly that

$$(3.25)\quad \int_a^b (u_2(t),\phi(t)' + (A(t)^* + E_2(t)^{*\prime})\phi(t))\,dt + \int_a^b (E_2(t)'u_1(t),\phi(t))dt + \int_a^b (E_2(t)f(t),\phi(t))\,dt - (u_b,\phi(b)) = 0.$$

Finally set $u(t) = \Sigma_{i=1}^2 u_i(t)$. Then $E_i(t)u(t) = u_i(t)$ for $i = 1,2$. Since $E_1(t)' + E_2(t)' = 0$ for $t \in [a,b]$ it follows from (3.24) and (3.25) that (3.5) holds. Hence $u(t)$ is a weak solution of (3.2), (3.3), and (3.4).

DEFINITION 3.4. Let $[\alpha,\beta] \subseteq [a,b]$. Let Q_1, Q_2 be the bounded operators from $C[\alpha,\beta]$ to $C[\alpha,\beta]$ that are defined by

$$(3.26)\qquad Q_1 g(t) = \int_\alpha^t W_1(t,\tau)g(\tau)d\tau$$

and

$$(3.27)\qquad Q_2 g(t) = \int_t^\beta W_2(t,\tau)g(\tau)d\tau$$

respectively. Let $R_1 = Q_2Q_1$ and set $R_2 = Q_1Q_2$.

THEOREM 3.5. *Suppose that $A(t)$ satisfies Conditions I, II, and III. There exists a $\delta > 0$ such that for every $[\alpha,\beta] \subset [a,b]$ that satisfies $(\beta-\alpha)\,\delta < 1$ the two-point problem*

$$(3.28)\qquad \frac{du}{dt} = A(t)u(t) + f(t),$$

$$(3.29)\qquad E_1(\alpha)u(\alpha) = u_\alpha,$$

$$(3.30)\qquad E_2(\beta)u(\beta) = u_\beta$$

has a unique weak solution for every $f \in C[\alpha,\beta]$, $u_\alpha \in E_1(\alpha)X$ and $u_\beta \in E_2(\beta)X$. There exists a constant C such that for every solution $u(t)$ of (3.28), (3.29), *and* (3.30) *and $t \in [\alpha,\beta]$ we have $\| u(t) \| \leqq C(\| u_\alpha \| + \| u_\beta \| + \max_{t\in[\alpha,\beta]}\| f(t) \|)$.*

PROOF. Let

$$C_0 = \max_{i=1,2}\left(\max_{a\leqq\tau\leqq t\leqq b}\| W_1(t,\tau)\|,\ \max_{a\leqq t\leqq\tau\leqq b}\| W_2(t,\tau)\|\right).$$

For every $g(t) \in C[\alpha,\beta]$ and for $i = 1,2$ we have

$$(3.31)\qquad | R_i g(t) |_0 \leqq (C_0(\beta-\alpha))^2 \,| g(t) |_0.$$

Let

$$g_1(t) = K_1(t,\alpha)u_\alpha + \int_\alpha^t K_1(t,\tau)E_1(\tau)f(\tau)d\tau \tag{3.32}$$

and set

$$g_2(t) = K_2(t,\beta)u_\beta + \int_t^\beta K_2(t,\tau)E_2(\tau)f(\tau)d\tau. \tag{3.33}$$

Suppose that $C_0(\beta - \alpha) < 1$. Then the system of equations

$$u_1(t) = g_1(t) + Q_1u_2(t), \tag{3.34}$$

$$u_2(t) = g_2(t) + Q_2u_1(t) \tag{3.35}$$

has a unique solution that is given by

$$u_1(t) = \sum_{j=0}^{\infty} R_2^j g_1(t) + Q_1 \sum_{j=0}^{\infty} R_1^j g_2(t) \text{ and} \tag{3.36}$$

$$u_2(t) = Q_2 \sum_{j=0}^{\infty} R_2^j g_1(t) + \sum_{j=0}^{\infty} R_1^j g_2(t). \tag{3.37}$$

Lemma 3.3 ensures that the two-point problem (3.27), (3.28), and (3.29) has a unique solution that is given by

$$u(t) = (I + Q_2) \sum_{j=0}^{\infty} R_2^j g_1(t) + (I + Q_1) \sum_{j=0}^{\infty} R_1^j g_2(t). \tag{3.38}$$

LEMMA 3.6. *Let $u(t)$ be a weak solution of* (3.2), (3.3), *and* (3.4). *For $i = 1, 2$, set $u_i(t) = E_i(t)u(t)$. Let $[\alpha,\beta] \subseteq [a,b]$. Set*

$$g_1(t) = K_1(t,\alpha)u_1(\alpha) + \int_\alpha^t K_1(t,\tau)E_1(\tau)f(\tau)d\tau \tag{3.39}$$

and set

$$g_2(t) = K_2(t,\beta)u_2(\beta) + \int_t^\beta K_2(t,\tau)E_2(\tau)f(\tau)d\tau. \tag{3.40}$$

Then $u_1(t)$ and $u_2(t)$ satisfy the equations (3.34) *and* (3.35) *in $[\alpha,\beta]$.*

PROOF. Lemma 3.6 is an immediate consequence of Lemma 3.3 and of the relations $K_1(t,a) = K_1(t,\alpha)K_1(\alpha,a)$ and $K_2(t,b) = K_2(t,\beta)K_2(\beta,b)$.

THEOREM 3.7. *Suppose that $A(t)$ satisfies Conditions I, II, and III. Assume in addition that for some positive α, β, and γ and for $i = 1,2$,*

$$L_i(t)^{-1} \in C_\alpha^1([a,b], B(X,X)), \quad E_i(t) \in C_\beta^1([a,b], B(X,X)),$$

and $f(t) \in C_\gamma([a,b], B(X,X))$. Then every weak solution of (3.2), (3.3), *and* (3.4) *is a strict solution.*

PROOF. For $a \leqq \tau \leqq t \leqq b$ let $S_1(t,\tau) = K_1(t,\tau) - \exp((t-\tau)L_1(t))$. It follows from the assumptions of the present theorem and from the results of [6] that there exist positive constants C and δ such that

$$\left\| \frac{\partial}{\partial t} \exp((t-\tau)L_1(t)) \right\| \leqq C(t-\tau)^{-1}, \tag{3.41}$$

$$\left\| \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \tau} \right) \exp((t-\tau)L_1(t)) \right\| \leqq C, \text{ and} \tag{3.42}$$

$$\left\| \frac{\partial}{\partial t} S_1(t,\tau) \right\| \leqq C(t-\tau)^{-1+\delta}. \tag{3.43}$$

Set $P_1(t) = E_1(t)E_1'(t)$. Let $[\alpha,\beta] \subseteq [a,b]$ and suppose that $h(t) \in C^1[\alpha,\beta]$. Then $Q_1 h(t) \in C^1[\alpha,\beta]$ and

$$\begin{aligned} \frac{d}{dt} Q_1 h(t) = & \int_\alpha^t \frac{\partial}{\partial t} S_1(t,\tau)P_1(\tau)h(\tau)d\tau \\ & + \int_\alpha^t \frac{\partial}{\partial t} \exp((t-\tau)L_1(t))(P_1(\tau) - P_1(t))h(\tau)d\tau \\ & + \int_\alpha^t \frac{\partial}{\partial t} \exp((t-\tau)L_1(t))P_1(t)(h(\tau) - h(t))d\tau \\ & + \int_\alpha^t \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \tau} \right) \exp((t-\tau)L_1(t))d\tau\, P_1(t)h(t) + \exp((t-\alpha)L_1(t))P_1(t)h(t). \end{aligned} \tag{3.44}$$

Let

$$M_1 = \sup_{a \leqq \tau < t \leqq b} (t-\tau) \left\| \frac{\partial}{\partial t} \exp((t-\tau)L_1(t))E_1(t)E_1'(t) \right\|.$$

It follows from (3.41), (3.42), and (3.43) that there exists a constant C such that for $i = 1$ we have

$$\left| Q_i h(t) \right|_1 \leqq C \left| h(t) \right|_0 + M_i(\beta - \alpha) \left| h(t) \right|_1. \tag{3.45}$$

Similarly there exist constants M_2 and C such that (3.42) holds for $i = 2$. Let $C_1 = \max_{i=1,2} M_i$. Then there exists a C such that for $i = 1,2$

$$\left| R_i h(t) \right|_1 \leqq C \left| h(t) \right|_0 + (C_1(\beta - \alpha))^2 \left| h(t) \right|_1. \tag{3.46}$$

Let $u(t)$ be a weak solution of (3.2), (3.3), and (3.4). For $i = 1,2$ set $u_i(t) = E_i(t)u(t)$. Define the function $g_1(t)$ and $g_2(t)$ by (3.39) and (3.40) respectively. Observe that there exists a C such that $\| (d/dt) g_1(t) \| \leqq C(t-\alpha)^{-1}$. Arguments similar to those used in the derivation of (3.45) ensure the existence of a C such that $\| (d/dt) Q_2 g_1(t) \| \leqq C \ln(t-\alpha)$ and $R_2 g_1(t) \in C^1[\alpha,\beta]$. Similarly $R_1 g_2(t)$

$\in C^1[\alpha,\beta]$. Suppose that for $i = 0,1$ $C_i(\beta-\alpha) < 1$. Here C_0 and C_1 are the constants appearing in the right-hand side of (3.31) and (3.46) respectively. Estimates (3.31) and (3.46) guarantee that $\sum_{j=1}^{\infty} R_2^j g_1(t) \in C^1[\alpha,\beta]$ and that $\sum_{j=1}^{\infty} R_1^j g_2(t) \in C^1[\alpha,\beta]$. Lemma 3.6 implies that $u_1(t)$ and $u_2(t)$ are given by (3.36) and (3.37) respectively. Since $g_1(t)$, $Q_2 g_1(t)$, $g_2(t)$, and $Q_1 g_2(t)$ belong to $C^1(\alpha,\ \beta)$ also $u_i(t) \in C^1(\alpha,\beta)$ for $i = 1,2$. Consequently $u(t) \in C^1(a,\ b)$. It follows from Lemma 3.6 that $u_1(t)$ is the weak solution of (3.11) with initial value equal to $u_1(\alpha)$. Since $u_2(t) \in C^1[\alpha,\beta]$, the above-mentioned results of [7] and the assumptions of the present theorem guarantee that $u_1(t)$ is a strict solution of (3.11). Consequently $u_1(t) \in D(A(t)E_1(t))$ for $t \in (\alpha,\beta]$. Since $u_1(t) = E_1(t)u(t)$, $u_1(t) \in D(A(t))$ for $t \in (\alpha,\beta]$. Similarly $u_2(t) \in D(A(t))$ for $t \in [\alpha,\beta)$. Hence $u(t) \in D(A(t))$ for $t \in (a,b)$ and $u(t)$ is a strict solution of (3.2), (3.3), and (3.4).

Let $A(x,D)$ be an $l \times l$ system of differential operators that is elliptic of order ω in $\bar{G}$ with coefficients that are infinitely differentiable in $\bar{G}$. Consider boundary operators $B_j(x,D)$, $j = 1,\cdots,\frac{1}{2}\omega l$, such that $B_j(x,D)$ is a $1 \times l$ system of differential operators of order $\omega_j < \omega$ with coefficients that are infinitely differentiable in $\bar{G}$. Denote by $H^{\omega,p}(G,\{B_j\})$ the completion of the set $\{u;\, u \in C^\infty(\bar{G}),\ B_j(x,D)u = 0$ on ∂G for $j=1,\cdots,\frac{1}{2}\omega l\}$ in $H^{\omega,p}(G)$. Let A_B^p be the unbounded linear operator in $H^{0,p}(G)$ such that $D(A_B^p) = H^{\omega,p}(G,\{B_j\})$ and $A_B^p u = A(x,D)u$ for $u \in D(A_B^p)$.

It is proved in [3] that if $A(x,D)$ and $B_j(x,D)$, $j=1,\cdots,\frac{1}{2}\omega l$, satisfy Agmon's conditions (see [1] and [9]) on the rays $l_{\frac{1}{2}\pi}$ and $l_{-\frac{1}{2}\pi}$ then there exist bounded projections E_1 and E_2 in $H^{0,p}(G)$ such that $A_B^p E_1$ and $-A_B^p E_2$ are infinitesimal generators of analytic semigroups and A_B^p is completely reduced by the direct sum decomposition

$$H^{0,p}(G) = \sum_{i=1}^{2} \oplus E_i H^{0,p}(G).$$

Regularity properties of $E_i(t)$ and $A_B^p(t)E_i(t)$, $i = 1,2$, are investigated in [4] for $A(t,x,D)$ and $B_j(t,x,D)$, $j = 1,\cdots,\frac{1}{2}\omega l$ dependent on t.

THEOREM 3.8. *For $t \in [a,b]$ let $A(t,x,D)$ be an $l \times l$ differential system that is elliptic of order ω in $\bar{G}$. Suppose that for $t \in \bar{G}$ the boundary operator $B_j(t,x,D)$, $j = 1,\cdots,\frac{1}{2}\omega l$, is of order $\omega_j < \omega$ in $\bar{G}$. Let $a(t,x)$ be any of the coefficients of $A(t,x,D)$ or of $B_j(t,x,D)$, $j = 1,\cdots,\frac{1}{2}\omega l$.*

(i) *Assume that for every multi-index α, $\partial^\alpha/\partial x^\alpha a(t,x)$ and $\partial/\partial t\ \partial^\alpha/\partial x^\alpha a(t,x)$ exist and are continuous in $[a,\ b] \times \bar{G}$. For every $t_0 \in [a,b]$ there exists an $r > 0$ such that $A_B^p(t)$ satisfies the assumptions of Theorem 3.5 in*

$$[a', b'] = [a, b] \cap \{t; |t - t_0| \leqq r\}.$$

(ii) *If in addition to the assumptions of (i) for every multi-index* α,

$$\frac{\partial}{\partial t} \frac{\partial^\alpha}{\partial x^\alpha} a(t, x)$$

is Holder continuous with respect to t in $[a, b]$, *uniformly with respect to x in* $\bar{G}$, *then* $A_B^p(t)$ *satisfies in* $[a', b']$ *the assumptions of Theorem* 3.7.

PROOF. The proof of [4, Th. 4.8] ensures the validity of (i) and (ii) provided that for every multi-index α there exist constants H_0 and H such that for $(t, x) \in [a, b]$, $\bar{G}$ and $n = 0, 1, \cdots$, we have $|(\partial^n/\partial t^n)(\partial^\alpha/\partial x^\alpha) a(t, x)| \leqq H_0 H^n M_n$ and M_n, $n = 0, 1, \cdots$, are suitably chosen constants. The proof of the present theorem is obtained from the proof of [4, Th. 4.8] with obvious modifications.

Let $0 < \theta < \frac{1}{2}\pi$ and write $t > \tau \pmod{\theta}$ if $t \neq \tau$ and $|\arg(t - \tau)| < \theta$. $t \geqq \tau \pmod{\theta}$ if $t > \tau \pmod{\theta}$ or $t = \tau$. Let $0 < \theta_1 < \frac{1}{2}\pi$ for $i = 1, 2$. Given a complex, convex neighborhood O of an interval $[a, b]$, set

$$O([a, b], \theta_1, \theta_2) = O \cap \{t; t > a \ (\mathrm{mod}\, \theta_1),\ b > t \ (\mathrm{mod}\, \theta_2)\}.$$

Let O be a convex, complex neighborhood of $[a, b]$. For $t \in O$ let $L(t)$ be a closed and densely defined linear operator in X. Assume that the following conditions are satisfied:

(i)′ For $t \in O$, the resolvent set of $L(t)$ contains the closed sector

$$\Gamma(-\tfrac{1}{2}\pi - \theta,\ \tfrac{1}{2}\pi + \theta),\ 0 < \theta < \tfrac{1}{2}\pi.$$

(ii)′ $L(t)^{-1}$ is analytic in O.

(iii)′ There exists a constant M such that for $\lambda \in \Gamma(-\frac{1}{2}\pi - \theta,\ \frac{1}{2}\theta\pi + \theta)$, and $t \in O$ we have

$$|(\lambda - L(t))^{-1}| \leqq M/|\lambda|. \tag{3.47}$$

It is shown in [8] that the evolution operator $U(t, \tau)$ of $L(t)$ has a continuation that is analytic for t, τ such that $t, \tau \in O$ and $t > \tau \pmod{\theta}$ and is strongly continuous for $t, \tau \in O$ such that $t \geqq \tau \pmod{\theta}$. Note that if $B(t) \in B(X, X)$ for $t \in O$ and $B(t)$ is analytic in O then there exists a complex μ such that $L(t) + B(t) + \mu I$ satisfies (i)′, (ii)′, and (iii)′.

THEOREM 3.9. *For* $t \in O$ *let* $A(t)$ *be a closed and densely defined linear operator in* X. *Assume that for* $t \in O$ *and* $i = 1, 2$, $E_i(t)$ *is a bounded projection and that* $A(t)$ *is completely reduced by the direct sum decomposition*

$$X = \sum_{i=1}^{2} \oplus E_i(t)X.$$

Suppose that for $i = 1,2$, $E_i(t)$ *is analytic in* O. *Assume that there exist constants* μ_i, $i = 1,2$ *such that* $L_1(t) = A(t)E_1(t) + \mu_1 I$ *and* $L_2(t) - (A(t)E_2(t) + \mu_2 I)$ *satisfy* (i)′, (ii)′, *and* (iii)′ *with* $\theta = \theta_1$ *and* $\theta = \theta_2$ *respectively. Let* $f(t)$ *be analytic in* O *and suppose that* $u(t)$ *is a solution of* (3.2), (3.3), *and* (3.4). *Then* $u(t)$ *has an extension that is analytic in* $O([a,b], \theta_1, \theta_2)$. *For* $t \in O([a,b], \theta_1, \theta_2)$ *we have* $u(t) \in D(A(t))$ *and* $(du/dt) - A(t)u(t) = f(t)$.

PROOF. For $i = 1,2$ and $t \in [a,b]$, set $u_i(t) = E_i(t)u(t)$. Let O' be a convex, complex neighborhood of $[a,b]$ such that the closure of O' is contained in O. Let $0 < \phi_i \leqq \theta_i$ for $i = 1,2$. Let $[\alpha,\beta] \subset [a,b]$. Define the functions $g_1(t)$ and $g_2(t)$ with the help of (3.39) and (3.40) respectively, and note that equations (3.34) and (3.35) are satisfied [5, Part 3, Lem. 2.1] and the above-mentioned results of [8] guarantee that for $j = 1,2$ for $i = 1,2$, and for $n = 0,1\cdots$, $R_i^n g_i(t)$ has an extension that is analytic in $O'([\alpha,\beta],\phi_1,\phi_2)$. Let ρ be the diameter of $O'([\alpha,\beta],\phi_1,\phi_2)$ and choose M so that $|W_1(t,\tau)| \leqq M$ for $t,\tau \in O'$, $t \geqq \tau$ (mod ϕ_1) and $|W_2(t,\tau)| \leqq M$ for $t,\tau \in O'$, $\tau \geqq t$ (mod ϕ_2). Then there exists a C such that for $i,j = 1,2$, for $n = 0,1,\cdots$, and for $t \in O'([\alpha,\beta],\phi_1,\phi_2)$, we have $|R_i^n g_j(t)| \leqq C(M\rho)^{2n}$. Let $M\rho < 1$. Then for $t \in [\alpha,\beta]$, $u_1(t)$ and $u_2(t)$ are given by the right-hand side of (3.36) and (3.37) respectively and have analytic extensions to $O'([\alpha,\beta], \phi_1,\phi_2)$. The last observation implies that $u(t)$ has an analytic extension to $O'([\alpha,\beta],\phi_1,\phi_2)$. Hence there exists an $h > 0$ such that $u(t)$ has an analytic extension to

$$O'([a,b], \phi_1,\phi_2) \cap \{t; |I_m t| < h\}.$$

For $i = 1,2$ and $t \in O'([a,b], \phi_1,\phi_2) \cap \{t; |I_m t| < h\}$ set $u_i(t) = E_i(t)u(t)$. Let $I = O'([a,b],\phi_1,\phi_2) \cap \{t; I_m t = \frac{1}{2}h)$ and let $[\alpha',\beta']$ be a subinterval of I. Then $[\alpha',\beta'] = O'([\alpha,\beta],\phi_1,\phi_2) \cap \{t; I_m t = \frac{1}{2}h)$ and $[\alpha,\beta] \subset [a,b]$. Lemma 3.6 and the relations $K_1(t,\alpha) = K_1(t,\alpha')K_1(\alpha',\alpha)$ and $K_2(t,\beta) = K_2(t,\beta')K_2(\beta',\beta)$ that hold for $t \in [\alpha',\beta']$ and the unique continuation property of analytic functions imply that for $t \in [\alpha',\beta']$ we have

$$u_1(t) = K_1(t,\alpha')u_1(\alpha') + \int_{\alpha'}^{t} K_1(t,\tau)E_1(\tau)f(\tau)d\tau + \int_{\alpha'}^{t} W_1(t,\tau)u_2(\tau)d\tau, \tag{3.48}$$

$$u_2(t) = K_2(t,\beta')u_2(\beta') + \int_{t}^{\beta'} K_2(t,\tau)E_2(\tau)f(\tau)d\tau + \int_{t}^{\beta'} W_2(t,\tau)u_1(\tau)d\tau. \tag{3.49}$$

Repeating the previous arguments we conclude that $u(t)$ is analytic in

$$O'([a,b],\phi_1,\phi_2).$$

Hence $u(t)$ is analytic in $O([a,b],\phi_1,\phi_2)$.

For $t\in O([a,b],\theta_1,\theta_2)$ and $i=1,2$, set $u_i(t)=E_i(t)u(t)$. Note that if $L(t)$ satisfies Theorem 3.8 (i)′, (ii)′, and (iii)′ then for every compact subset K of O there exists a constant C_1 such that for $t\in K$ and $\lambda\in\Gamma(-\frac{1}{2}\pi-\theta,\ \frac{1}{2}\pi+\theta)$ we have $\|(\partial/\partial t)(\lambda-A(t)^{-1})\|\leqq C_1/|\lambda|$. (See [8].) This observation, Lemma 3.5, Theorem 3.7, and the validity of (3.48) and (3.49) for every horizonta lsubinterval $[\alpha',\beta']$ of $O([a,b],\theta_1,\theta_2)$ ensure that for $t\in O([a,b],\theta_1,\theta_2)$ we have $u(t)\in D(A(t))$ and $(du/dt)-A(t)u(t)=f(t)$.

THEOREM 3.10. *Let O be a convex, complex neighborhood of $[a,b]$. For $t\in O$ let $A(t,x,D)$ be an $l\times l$ elliptic differential system of order ω in $\bar{G}$. Suppose that for $t\in O$ the boundary operator $B_j(t,x,D), j=1,\cdots,\frac{1}{2}\omega l$ is of order $\omega_j<\omega$ in $\bar{G}$. Denote by $a(t,x)$ any of the coefficients of $A(t,x,D)$ or of $B_j(t,x,D), j=1,\cdots,\frac{1}{2}\omega l$. Assume that $a(t,x)$ has derivatives of all orders with respect to x that are continuous in $O\times\bar{G}$. Suppose that for $x\in\bar{G}$, $a(t,x)$ is analytic in O. Let $0<\theta_i<\frac{1}{2}\pi$, $i=1,2$, and suppose that for $t\in O$ and $\theta\in[\frac{1}{2}\pi-\theta_2,\ \frac{1}{2}\pi+\theta_1]\cup[-\frac{1}{2}\pi-\theta_1,-\frac{1}{2}\pi+\theta_2]$, $A(t,x,D)$ and $B_j(t,x,D)$, $j=1,\cdots,\frac{1}{2}\omega l$, satisfy Agmon's conditions on l_θ. Let $f(t)$ be analytic in O. Assume that*

$$u(t)\in C[a,b]\cap C^1(a,b)$$

and that for $t\in(a,b)$, $u(t)\in D(A_B^p(t))$ and $(du/dt)-A_B^p(t)u(t)=f(t)$ holds. Then $u(t)$ has an extension that is analytic in $O([a,b],\theta_1,\theta_2)$. For $t\in O([a,b],\theta_1,\theta_2)$ we have $u(t)\in D(A_B^p(t)$ and $(du/dt)-A_B^p(t)u(t)=f(t)$.

PROOF. Arguing as in the proof of [4, Th. 4.8] we conclude that for every $t_0\in[a,b]$ there exists an interval $[\alpha,\beta]$ such that $t_0\in(\alpha,\beta)$ and the assumptions of Theorem 3.9 hold in $O([\alpha,\beta],\theta_1,\theta_2)$. Hence there exists an $h>0$ such that the assertions of the present theorem hold in $O([a,b],\theta_1,\theta_2)\cap\{t;|\mathrm{Im}\,t|\leqq h\}$. Observing that $(du/dt)-A_B^p(t)u(t)=f(t)$ for $t\in O([a,b],\theta_1,\theta_2)\cap\{t;\mathrm{Im}\,t=h\}$ we complete the proof by repetition of the previous arguments.

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